# Strocchi's Quantum Mechanics: An alternative formulation to the dominant one?

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*Abstract*: At first glance, Strocchi's formulation presents several characteristic features of a theory whose two choices are the alternative ones to the choices of the paradigmatic formulation: i) Its organization starts from not axioms, but an operative basis and it is aimed to solve a problem (i.e. the indeterminacy); moreover, it argues through both doubly negated propositions and an *ad absurdum* proof; ii) It put, before the geometry, a polynomial algebra of bounded operators; which may pertain to constructive Mathematics. Eventually, it obtains the symmetries. However one has to solve several problems in order to accurately re-construct this formulation according to the two alternative choices. I conclude that rather than an alternative to the paradigmatic formulation, Strocchi's represents a very interesting divergence from it.

*Keywords*: Quantum Mechanics, C*-algebra approach, Strocchi's formulation, Two dichotomies, Constructive Mathematics, Non-classical Logic

## 1. Strocchi's Axiomatic of the paradigmatic formulation and his criticisms to it

Segal (1947) has suggested a foundation of Quantum Mechanics (QM) on an algebraic approach of functional analysis; it is independent from the space-time variables or any other geometrical representation, as instead a Hilbert space is. By defining an algebra of the observables, it exploits Gelfand-Naimark theorem in order to faithfully represent this algebra into Hilbert space and hence to obtain the Schrödinger representation of QM. In the 70's Emch (1984) has reiterated this formulation and improved it. Recently, Strocchi improved it much more.

First, Strocchi has suggested an axiomatic of the paradigmatic Dirac-von Neumann's formulation of QM (= DvNQM).

> *Axiom I. States*. The states are represented by rays (or matrices) in a Hilbert space […]. *Axiom II.* Observables. The observables of a quantum mechanical systems, i.e. the quantities which can be measured, are described by the set of bounded self-adjoined operators in a Hilbert space H […] *Axiom III.* Expectations [of an experiment applying an operator to a state ω] is given by the Hilbert space matrix element $<A\omega> = (\Psi\omega, A\ \Psi\omega)$ […] *Axiom IV.* Dirac canonical quantization. The operators which describe the canonical coordinates $q_i$ and moment $p_i$, $i = 1,\dots s$ of a quantum system of 2s degrees of freedom obey the canonical commutation relations. $[q_i, q_j] = 0 = [p_i, p_j]$. $[p_i, q_j] = I\ h/2\pi\ \delta_{ij}$ […] *Axiom V.* Schrödinger representation. The [previ-

> ous] commutation relations… are represented by the following operators in the Hilbert space $\mathcal{H} = L^2(R^s, dx)$: $q_i\ \psi(x) = x_i\ \psi(x)$; $p_j\psi(x) = ih/2\pi\ \partial\psi/\partial x_j\ (x)$ (Strocchi 2012, pp. 1-3).

To this formulation Strocchi addresses two basic criticisms. First, a weak linkage with the experimental basis of theoretical physics.

> The Dirac-von Neumann axioms provide a neat mathematical foundation of quantum mechanics, but their a priori justification is not very compelling, their main support, as stressed by Dirac, being the *a posteriori* success of the theory they lead to. The dramatic departure from the general philosophy and ideas of classical physics may explain the many attempts of obtaining quantum mechanics by a deformation of classical mechanics or by the so-called geometric quantization. Thus, a more argued motivation on the basis of physical considerations is desirable (Strocchi 2012, p. 3).

## 2. The operative starting point of Strocchi's formulation

Strocchi declares in the following terms his starting point for constructing a new theory:

> The discussion of the principles of QM gets greatly simplified, from a conceptual point of view, if one first clarifies what are the [physical] objects of the [subsequent] mathematical formulation (Strocchi 2012, p. 3).

These objects are essentially the physical apparatuses and the physical operations, which neither Segal nor Emch discussed. Hence, as a first step, Strocchi replaces for previous Axioms I and II of DvNQM a detailed analysis of the experimental basis of theoretical physics so that he suggests a clearly operative support to the notions of operator A, state ω and expectation. In particular, he relates the boundedness of all operators of Segal's algebraic approach to the experimental constraint of all physical measurements, i.e. to give finite results only.

## 3. Justifying a C*-algebra for QM: Strocchi's Axiom A

However, his analysis meets some difficulties in operationally justifying the wanted C*-algebra.[1] He has to define the sum of disparate operators; potential and kinetic energy is a classical sum; but e.g. mass and force, or temperature and volume both appear as idealistic operations (the product of two operators is not defined in order to later take in account the non commutativity). Strocchi honestly admits that his work is only partially successful. Thus, we skip to his unique Axiom

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[1] A C*-algebra is a Banach algebra on a complex field, together an involution * with the property $\|A^*A\| = \|A^2\|$ ·A Banach algebra is a linear associative algebra over the field C of the complex numbers with a norm ‖ ‖. A norm is essentially a bound; more precisely, it is a function which assigns a strictly positive length or size to each vector in a vector space.

> The following Axiom [...] partly goes beyond the implications of the operational analysis discussed so far; however, in our opinion, it represents a more physically motivated alternative to Dirac-von Neumann axiom II. All the preceding discussions and arguments are meant to provide a[possible] physical justification of such an axiom and are completely summarized and superseded by it. An indirect justification of it as a property of the description of a general physical systems is that it is satisfied by *both CM and QM*.
>
> *Axiom A*. The observables generate a [polynomial] C*-algebra $\mathcal{A}$, with identity [...]; the *states* which by eq. (2.1) define positive linear functionals on the Algebras $\mathcal{A}_A \subset$ A, for any observable A, separate such algebras in the sense of eq. (2.6) and extend to positive linear functional on $\mathcal{A}$ (Strocchi 2012, p. 6).

As an important consequence of this axiom, the ambiguity about hermiticity and self-adjointness of the operators in DvNQM[2] is cancelled because for bounded operators hermiticity implies self-adjointness (Strocchi 2012, p. 2). This result solves the question, discussed by (Emch 1984, pp. 378-379), why to bound the C*-algebra to self-adjoint operators only.

## 4. Relationship of C*-algebra with Hilbert space

Then Strocchi exploits the mathematical advancement by the Gelfand-Naimark-Segal theorem (GNS) for recovering the Hilbert space and hence the geometrical description of a physical system. This application of the C*-algebra replaces the above Axiom III of DvNQM.

> From the point of view of general philosophy, the picture emerging from the Gelfand theory of abelian C*-algebras has far reaching consequences and it leads to a rather drastic change of perspective [in theoretical physics]. In the standard description of a physical system the geometry comes first: one first specifies the coordinate space (more generally a manifold or a Hausdorff topological space), which yields the geometrical description of the system, and then one considers the abelian algebra of continuous functions on that space. By the Gelfand theory [instead] the relation can be completely reversed: one may start from [an algebra, i.e.] the abstract abelian C*-algebra, which in the physical applications may be the abstract characterization of the observables, in the sense that it encodes the relations between the physical quantities of the system, and then one reconstructs the Hausdorff space such that the given C*-algebra [with identity] can be seen as the C*-algebra of continuous functions on it. In this perspective, one may say that the algebra comes first, the geometry comes later […] (Strocchi 2010, p. 15).

[2] An operator A is adjoint if there is A* such that (Ax, y) = (x, A*y), where * is the involution. It is self-adjoint if A = A*.

> In conclusion, from the above considerations it follows that the right language for the mathematical description of quantum systems is the theory of (non-abelian) C*-algebras and as such the mathematical structure of quantum mechanics can be viewed as a chapter of that theory (Strocchi 2010, p. 42).

## 5. The representation of the principle of indeterminacy and Dirac's quantization

Strocchi second criticism to the axiomatic of DvNQM is the obscurity about the separation mark between classical mechanics and QM. He underlines that nothing obstructs to represent a classical system inside Hilbert space. The only difference is that

> Classical mechanics results in a Hilbertian description which is equivalent to one in terms of an algebra of functions, whereas this kind of algebra is impossible when the observables do not commute [since two mutually interfering variables cannot be governed by the notion of a function] (Strocchi 2012, p. 9).

The quantum characterization enters through the *Axiom IV*, concerning the non-commutativity of the two conjugate observable defining a states. Actually, this quantum/classical distinction was blurred for a long time because the status of the principle of indeterminacy was unclear to most physicians. In 1947 Segal had still to write that he had

> To confute the view that the indeterminacy principle is a reflection of an unduly complex formulation of Quantum mechanics and to [strength] the view that the principle is quite intrinsic in physics, or in an empirical science based on quantitative measurement (Segal 1947, p. 931).

About non-commutativity first Strocchi remarks that the usual mathematical relations are not valid for finitely measurable operators, essentially because a sharp measurement of one observable (e.g. $\Delta p = 0$ exactly) ought to have in correspondence an infinite value of the other observable; yet, this value cannot be operationally obtained (Strocchi 2012, p. 8). Hence, he evaluates as insufficient Born's and Heisenberg's experimental justifications for these relations. Rather, he advances reasons of experimental methodology for suggesting a new mathematical version of them (called by him "complementarity relations"):

$$\Delta_\omega(\mathrm{A}) + \Delta_\omega(\mathrm{B}) \geq \mathrm{C} > 0 \text{ for all } \omega$$

where $\Delta$ is the mean square deviation. Notice that this relation is not the mere logarithm of the previous one because it may differ at the infinity points.

> This provides a precise *operational and mathematical formulation* of complementarity with the advantage, w.r.t. the Heisenberg uncertainty relations, of being meaningful and therefore testable for operationally defined observables, necessarily represented by bounded operators [...] (Strocchi 2012, p. 8).

In particular, he proves that his version is more effective than Heisenberg's in the case of the two components $s_1$ and $s_3$ of momenta of spin ½ (Strocchi 2012, p. 9).
Second, Strocchi recalls the insufficient justification of Dirac canonical quantization, obtained by a mere analogy with classical Mechanics and not always valid. He re-formu-

lates it according to an algebraic comprehensive approach of Classical mechanics and QM. By starting from a free C*-algebra,[3] he directly obtains two cases of quantization through a dichotomic variable $Z$, whose values $Z = 0$ and $Z = i\text{h}/2\pi$ respectively correspond to classical Mechanics and QM; moreover, it proves that no other cases are possible beyond the above two. This result about quantization replaces previous Axiom IV of DvNQM (Strocchi 2010, pp. 10-11).

## 6. Schroedinger representation. Symmetries

Axiom V of DvNQM gives the Schrödinger representation inside Hilbert space. In SQM

> Schrödinger QM follows from the von Neumann uniqueness theorem (Strocchi 2008, p. 4th of the cover).

through the canonical commutators relations. SQM includes the symmetries too, as it is shown in the case of the dynamics in a one-parameter group of *-automorphisms of $\mathcal{A}$. In order to take in account the unboundedness of the operators, in this case he defines (rather than Heisenberg algebra) the Weyl algebra of the two variables, $p$ and $q$, defining the state of a particle.

> For finite degrees of freedom, the Weyl algebra codifies the experimental limitations on the measurements of position and momentum (Heisenberg uncertainty relations) […] (Strocchi 2008, p. 4th of the cover).

And the symmetries easily follow from von Neumann theorem on the uniqueness of all regular, irreducible representations of Weyl C*-algebra.

In sum, through the technique of the representations of C*-algebra, or better the Axiom A only, he has obtained a complete formulation of both QM and classical Mechanics. At last, Strocchi summarizes his formulation through the following features:

> In conclusion, the operational definition of states and observables motivates the physical principle or axiom that, quite generally the observables of a physical (not necessarily quantum mechanical) system generate a C*-algebra. The Hilbert space realization of states and observables (Dirac-von Neumann Axioms I-III) is then [obtained as] a mathematical result. The existence of observables which satisfy the operationally defined complementarity relations implies that the algebra of observables is not Abelian and it marks the difference between CM and QM. Thus, for a quantum mechanical system the Poisson algebra generated by the canonical variables [i.e. the algebraic-differential relationships between the variables] cannot be represented by commuting operators [owing to the indetermination relationships] and actually canonical quantization (*Axiom IV*) follows from general geometrical structures. The Schrödinger representation (*Axiom V*) is selected by the general properties of irreducibility and regularity. The general setting discussed so far may then pro-

[3] A free algebra is the non commutative analogue of a polynomial ring since its elements may be described as "polynomials" with non-commuting variables.

> vide a more economical and physically motivated alternative to the Dirac-von Neumann axioms for the foundation of quantum mechanics (Strocchi 2012, p. 12).

## 7. Strocchi's formulation as a PI theory. The lacking characteristic features

Hilbert space of (square summable) functions of calculus clearly represents the AI choice. Instead, Segal's suggestion, being based on an algebraic approach, whose historical tradition relies on constructive mathematical tools, promises an entirely new foundation of QM. As a fact, Strocchi's works suggest an at all new formulation of QM which before put the algebra and later the geometry, as also Heisenberg's formulation of QM did. With respect to the expectations of a quantum measurement this approach deals first with the operators, rather that the states, as Hilbert space does; this choice leads to stress the experimental characteristic feature of the entire formulation, in particular Heisenberg's principle, which Strocchi represents according to a more appropriate mathematical formula which avoids infinities. At last, its theoretical development obtains through Weyl algebra the mathematical technique of the PI&PO theories. In the literature on the QM that I know, I have found no one formulation presenting these merits; only Weyl formulation presents symmetries, yet introduced in an approximate way. For these reasons one may suppose that Segal's tradition represents an unaware and incomplete attempt elicited by many scholars to achieve a formulation of QM which is based on constructive mathematics. In particular, SQM looks as a good basis for searching a constructive (PI) formulation of QM.

In view of improving it as an entirely constructive formulation one has to discover the constructive counter-parts of the following steps of this theory:

1) Segal's tradition assumes the boundedness of each physical variable. This assumption is necessary in order to obtain a C*-algebra of the observables; it assures both the hermiticity of all operators and moreover the solutions of all relevant, differential equations (Pour-El Richards 1989). Strocchi tries to justify this thesis of boundedness through an operational analysis of experimental physics. In my opinion this thesis remains as questionable on an epistemological basis. This objection to his thesis challenges Strocchi's criticisms to the dominant formulation.
2) The mathematical definition of a C*-algebra. It there exists, provided that one accepts the apartness definition (See Bishop Bridges 1986, chp. 7, p. 157; Takamura 2005, p. 81).
3) GN theorem. In the case of Abelian algebras; its constructive counter-part was obtained by (Bridges 1979, sect. 6.7; Tanaka 2005, p. 289) through a slightly different notion of norm. Instead, in the case of a non-Abelian algebra, that is necessary for QM, to find a solution seems hopeless (Bridges 2017).

4) The argument of the *ad absurdum* proof (AAP) in the next sect. requires to derive from a polynomial C*-algebra a C*-algebra of general functions. Open problem.
5) The Dirac quantization through Strocchi's free algebra. Open problem.
6) The introduction of both Heisenberg and Weyl algebras and groups in the case of a finite number of observables. Open problem.
7) Von Neumann theorem (all regular irreducible representation of Weyl C-*algebras are unitarily equivalent). Open problem.

The difficulties presented by the above unsolved problems seem formidable.

## 8. Strocchi's formulation as a PO theory. The lacking characteristic features

An accurate inspection of SQM shows that it shares several characteristic features of a PO (see Drago 2012).

1) First of all, he argues by means of the intuitionist logic, inside which the law of double negation fails. Indeed, he makes use of doubly negated propositions whose corresponding affirmative propositions lack of evidence or are false (DNPs). In the following, I will spent some space for listing the DNPs occurring in (Strocchi 2012):[4]

a) t is <u>impossible</u> to measure coherent superpositions of states belonging to <u>different</u> superselection sectors. [≠ one measures coherent superpositions of states inside a single sector] (Strocchi 2012, p. 2).
b) Thus, if two states defined by two apparently different preparation procedures yield the same results of measurements for all observables, i.e. expectations, from an experimental point of view they <u>cannot</u> be considered as physically <u>different</u> [≠ they are the same] ... [to be cont.ed].
c) [...] since there is <u>no</u> measurement which <u>distinguishes</u> them [≠ the results of all measurements are equal] (Strocchi 2012, p. 3).
d) Similarly [...] there is <u>no</u> available operational way to <u>distinguish</u> them [≠ all operations give the same result] (Strocchi 2012, p. 3).
e) […] the <u>in</u>-<u>evitable</u> <u>limitations</u> in the preparation of states and measurements of *A* in general <u>preclude</u> the possibility of obtaining sharp values of A, i.e. $\Delta_\omega(A) = 0$..[≠ the freedom of preparations [...] gives [...] sharp values of …] (Strocchi 2012, p. 8).
f) *Experimental principle* […] For any given observable A, one can correspondingly prepare states for which a sharp value may be approximated as well as one likes [Here the nature of DNP is given by the point underlined words; they are equivalent to "<u>beyond</u> any <u>bound</u>"; ≠ at the infinity] (Strocchi 2012, p. 8).

[4] In the following I will underline the negative words inside a DNP in order to make apparent its logical nature. Notice that the modal words are equivalent to a DNP (e.g. may: "it <u>not</u> <u>false</u> that it is the case that…" They will be point underlined.

> g) This means that it is impossible to have a direct [= non mediated] experimental check of the uncertainty relations [≠ one has a mediated experimental check of the uncertainty relations] ... [to be cont.ed]
> h) [...] since one only [= not otherwise ≠ surely] measures bounded functions of the position and the momentum (Strocchi 2012, p. 8).

A last proposition of this kind is presented by Strocchi when he introduces a crucial notion. Consistently with the PO model of a theory, he proceeds in a heuristic way in order to look for the mathematical version of the uncertainty relations. In addition, his main result (the proposition 2.8) is a DNP as it will be proved in the following. In a first time he suggests the new definition of complementarity through a negative word:

> Definition 2.7. Two observable A, B are called complementary if the following bound holds
>
> $\Delta$ (A) + $\Delta$ (B) > 0" (Strocchi 2012, p. 8)

Then he states the DNP:

> i) Proposition 2.8. If the above experimental principle holds, given a representation $\pi$ of A the existence of two observables $\pi$(A), $\pi$(B) which are complementary, implies that the C*-algebra A(A,B) generated by $\pi$(A), $\pi$(B) cannot be commutative [≠ two observables with A (A) + $\Delta$ (B)= 0 commute] (Strocchi 2012, p. 9).[5]

The given problem is not considered as solved without showing the relation between the old and the new notions. First, he relaxes the previous limitation of the observables to be represented by polynomial functions.

> The relation between complementarity and non-commutativity is easily displayed if one realizes that in each irreducible representation $\pi(\mathcal{A})$ of the algebra of observables one may enlarge the notion of observables by considering as observables the weak limits of any Abelian C*-subalgebra $\mathcal{B} \subset \pi(\mathcal{A})$. Technically, this amounts to consider the von Neumann algebra $\mathcal{B}^{w}$ generated by $\mathcal{B}$; one may show that the former contains all the spectral projections of the elements of B. In the Gelfand representation of the Abelian C*-algebra $\mathcal{B}$ by the set of continuous functions on the spectrum of $\mathcal{B}$, such weak limits correspond to the pointwise limits of the continuous functions. They are operationally defined by instruments whose outcomes yield the pointwise limits of the functions defined by the measurements of the elements of $\mathcal{B}$.
>
> This means that one recognizes as observables not only the polynomial functions of elements B belonging to $\mathcal{B}$ and therefore by norm closure the continuous functions of B, but also their pointwise limits (Strocchi 2012, p. 9).

2) Strocchi argues through an AAP. Indeed, the relationship between the two above relations is stated by means of an AAP, exactly the way of reasoning

[5] Notice that the second negative proposition is not a mere explanation of the first negative proposition, because they are different, physical the former one and mathematical the latter one.

of a PO theory. The argument can be summarized in the following way. By calling "complementarity of A,B" Cp and their "commutativity" Cm, he wants to prove that when Cp holds true then ¬Cm follows. He starts by negating the thesis, ¬¬Cm, which describes a situation where both π(A) and π(B) (according to a von Neumann's theorem) can be written as functions of C, i.e. in this case the C*-algebra is an algebra of functions. Hence, in this algebra the classical logic holds true, and thus ¬¬Cm → Cm. His arguing obtains that *Cm* → ¬*Cp*, i.e. the negation of the starting hypothesis, an absurd. Hence, it is not possible that *Cm* → *Cp*, or, ¬(*Cp* → ¬*Cm*), i.e. the new notion *Cp* surely grasps more content than the old notion *Cm*.[6]

3) He lucidly bases his theory on the problem of how our knowledge can overcome the unavoidable uncertainty of the measurements of two conjugate observables.

The main problem is the precise interpretation of the principle [of non commutativity of conjugate variables] in terms of unambiguous experimental operations and its precise mathematical formulation (Strocchi 2012, p. 15).

4) Yet, the above AAP concerns the relationships of the experimental basis of QM with DvNQM, not the conclusion of the theoretical development of SQM as solving this problem. Hence, one has to organize anew the original development of SQM by basing it on the above problem, at the cost to change some its parts. In fact, the actual starting point of his formal development of SQM is the Axiom A; Strocchi admits that it is not enough sufficiently supported by his "preliminary basic consideration" (Strocchi 2012, p. 6). However, nothing opposes to consider the Axiom A as a methodological principle in the aim at solving the above basic problem in suitable circumstances (see the similar L. Carnot's change of the common inertia principle; Drago1988). In such a case SQM is relying upon no more than the mathematical content of i.e. the polynomial C*-algebras of the observables. The boundedness postulate is then admissible, since an algebraic approach does not require the usual idealization of the experimental results by a theoretical physicist; usually, since wants to operate with real numbers and functions, the latter one extrapolates from a finite collection of experimental data a real function, including its points at infinity. Instead, a theorist following the algebraic approach can without problems assume the experimental data in their boudedness. That amounts to avoid the AI assumption on the experimental data for instead bounding the theoretical development to the PI. Moreover, to choose PO allows to start from a finite set of experimental data and hence to state a bound to all their values. These considerations solve the question 1) of previous section.

[6] Incidentally, in classical logic the proved formula *Cp* → ¬*Cm* is classically equivalent to ¬*Cp* v ¬*Cm* = ¬*Cm* v ¬*Cp* = *Cm* →¬*Cp*.

5) However, one has to suggest a theoretical development where one makes use of more DNPs than those used by Strocchi.
6) Moreover, one has to invent a chain of AAPs (including the previous AAP of SQM), concerning the resolution of the basic problem.
7) The resolution of the basic above problem is given by the Dirac quantization, that Strocchi obtains as a mathematical consequence of his C*-algebra. This suffices for closing the kinematics of QM; which through the GNS theorem includes Hilbert space.
8) At last, one has to apply to the conclusion of the final AAP the principle of sufficient reason for translating this conclusion in an affirmative proposition; from which one has then to obtain the symmetries.

In sum, in order to change SQM into a PI&PO theory one has to invent a great part of the wanted theory. The task is hard, but a priori not impossible.

All in all, although SQM shares several characteristic features of both choices PI and PO, at present time it is far from being an alternative theory to DvNQM, although its distance is the minimal one among the formulations I have already examined.

I conclude that rather than an alternative to the paradigmatic formulation, the present SQM represents a very interesting divergence from it.

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